Engineering Notes

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Optimal Intersatellite Transfers for On-Orbit Servicing Missions

E. H. Fallin III*
The Aerospace Corporation, El Segundo, Calif.

Nomenclature

A = symbol for a complex expression
a = ellipse semimajor axis, naut mile
B = symbol for a complex expression
b = symbol for a number less than one
c = symbol for a positive integer
d = symbol representing a differential
m = power to which an expression is raised
N = number of intersatellite transfer ellipse revolutions
n = number of intersatellite transfers
r = radius of apogee or perigee, naut mile
V = velocity, fps
$\Delta\lambda$ = phase (position) change in a circular orbit, deg
ΔT = total time allocated to transferring from the first to the last satellite, days or sec as required
ΔV = change in velocity, fps
Σ = symbol for summation
τ = period of an orbit, days or sec as required
μ = Earth's gravitational constant, naut mile^3/sec^2

Subscripts

i = ith transfer ellipse
j = jth transfer ellipse
k = kth transfer ellipse
T = total
1 = first
2 = second

Introduction

MISSIONS involving visits to several satellites located in the same circular orbit have recently been of interest to portions of the aerospace industry. Visits to several satellites are characteristic of on-orbit servicing missions, in which a service vehicle visits the satellites and removes and replaces malfunctioning equipment. These missions frequently involve service vehicles whose ΔV capability is limited. It is therefore of considerable interest to be able to conduct the intersatellite transfers in an optimal manner—that is, in a manner that minimizes the ΔV expended. Of course, taking a longer period of time to complete the transfers results in a lessening of the ΔV requirement whether the optimal allocation of velocity increments is chosen or not. What is being discussed here is the optimal allocation of velocity increments when there is a time constraint.

The problem that is addressed, then, is as follows: a) there are $(n+1)$ satellites serviced in the same circular orbit with specified central angle separations (n is the number of intersatellite transfers required); b) the total time allocated to transferring from the first to the last of the satellites is constrained to be no more than ΔT units of time; and c) given a) and b), it is required to find the distribution of transfer ellipses among the intersatellite transfers that minimizes the total ΔV requirement.

It is assumed for simplicity that all of the satellites visited are in the same plane. Each intersatellite transfer requires two velocity increments. The first initiates an elliptical orbit whose successive returns to circular orbit radius bring it closer to the objective satellite. An integral number of complete revolutions in the elliptical orbit is required. The second velocity increment circularizes the orbit at the objective satellite's location. The total ΔV requirement is the sum of all of the velocity requirements for all of the intersatellite transfers.

Derivation

Initial Considerations

The derivation begins with a simple definition of ΔV:

$$\Delta V = 2\,|\,V_2 - V\,| \tag{1}$$

It is assumed that each $\Delta V/2$ is, for practical purposes, applied instantaneously. A well-known expression for V is[1]

$$V = (\mu/r_1)^{1/2} \tag{2}$$

$$V_2 = V\left[\frac{2r_2}{r_1+r_2}\right]^{1/2} \tag{3}$$

$$\Delta V = 2V\,\left|\,\left(\frac{2}{1+r_1/r_2}\right)^{1/2} - 1\,\right| \tag{4}$$

Introduction of Central Angle Differences and Number of Revolutions

It is intended to transfer over each central angle difference, $\Delta\lambda_i$, in N_i revolutions. Then the central angle difference per revolution is $\Delta\lambda_i/N_i$, and it may be quickly shown that the period of the required ellipse is

$$\tau_i = \tau\left(1 - \frac{\Delta\lambda_i}{360\,N_i}\right) \qquad i=1,2,3,\ldots \tag{5}$$

$\Delta\lambda_i$ is positive when traveling in the direction of orbital motion and negative when traveling in the opposite direction. Another way of writing the period is

$$\tau = \frac{2\pi a^{3/2}}{\mu^{1/2}} \tag{6}$$

where the subscripts, i, have been omitted for simplicity. Combining Eqs. (5) and (6) results in

$$[1 + (r_2/r_1)] = 2[1 - (\Delta\lambda/360N)]^{2/3} \tag{7}$$

Since

$$\frac{2}{1+r_1/r_2} = 2 - \frac{2}{1+r_2/r_1} \tag{8}$$

combining Eqs. (7) and (8) results in

$$\Delta V = 2V\,\left|\,\left[2 - \left(1 - \frac{\Delta\lambda}{360\,N}\right)^{-2/3}\right]^{1/2} - 1\,\right| \tag{9}$$

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*Member of the Technical Staff, Shuttle and Integration Office. Member AIAA.

The objective of expressing ΔV in terms of $\Delta\lambda$ and N has thus been achieved. Alternative forms of Eq. (9) are given in the Appendix.

ΔV for More Than One Intersatellite Transfer

Equation (9) expresses the total velocity increment required to transfer between two satellite positions. When n transfers are required between $(n+1)$ satellite positions, the total velocity requirement can be written as

$$\Delta V_T = \sum_{i=1}^{n} \Delta V_i \tag{10}$$

where each of the ΔV_i is given by Eq. (9).

Use will now be made of the concept of finding a minimum (or maximum) by means of taking a differential and setting the differential equal to zero. It is helpful to make the following definitions:

$$A_i = [2 - (1 - \frac{\Delta\lambda_i}{360\, N_i})^{-2/3}] \tag{11}$$

$$B_i = (1 - \frac{\Delta\lambda_i}{360\, N_i}) \tag{12}$$

The differential of ΔV_T, using the definitions in Eqs. (11) and (12), is

$$d\Delta V_T = -\frac{V}{540} \sum_{i=1}^{n} | A_i{}^{-1/2} B_i{}^{-5/3} \frac{\Delta\lambda_i}{N_i{}^2} | \, dN_i \tag{13}$$

The requirement will be set that the total number of revolutions used in all transfers will be fixed at N_T. This constraint follows from the relationship[2] between the total time allocated to intersatellite transfers and the total number of revolutions. This relationship can be expressed by

$$N_T = \frac{\Delta T}{\tau} + \frac{\Sigma\Delta\lambda_i}{360} \tag{14}$$

ΔT can only take values that make N_T an integral number since an integral number of ellipses is always required for intersatellite transfers of the type being considered. Since N_T is defined as being constant, the sum of differentials of its constituent parts, the N_i, is zero. This can be written as

$$dN_1 = -dN_2 - dN_3 - \ldots - dN_n \tag{15}$$

Substitution of Eq. (15) in Eq. (13), setting $d\Delta V_T = 0$, and rearranging results in the following equation:

$$\{ | A_1{}^{-1/2} B_1{}^{-5/3} \frac{\Delta\lambda_1}{N_1{}^2} | - | A_2{}^{-1/2} B_2{}^{-5/3} \frac{\Delta\lambda_2}{N_2{}^2} | \} dN_2$$
$$+ \{ | A_1{}^{-1/2} B_1{}^{-5/3} \frac{\Delta\lambda_1}{N_1{}^2} | - | A_3{}^{-1/2} B_3{}^{-5/3} \frac{\Delta\lambda_3}{N_3{}^2} | \} dN_3 \cdots$$
$$+ \{ | A_1{}^{-1/2} B_1{}^{-5/3} \frac{\Delta\lambda_1}{N_1{}^2} | - | A_n{}^{-1/2} B_n{}^{-5/3} \frac{\Delta\lambda_n}{N_n{}^2} | \} dN_n = 0 \tag{16}$$

The nontrivial way in which the left-hand side of Eq. (16) can be zero is for each of the terms in brackets to be equal to zero. That this must be so follows from this line of reasoning: The dN_i of Eq. (16) are all nonzero and have $-dN_1$ as their sum, as shown in Eq. (15). Equation (16) must hold no matter what values the individual dN_i take as long as their sum remains constant at this value of $-dN_1$. This can only be true if each of the terms in brackets is identically equal to zero. Meeting this requirement can be expressed by $(n-1)$ equations of the following form:

$$| A_1{}^{-1/2} B_1{}^{-5/3} \frac{\Delta\lambda_1}{N_1{}^2} | = | A_i{}^{-1/2} B_i{}^{-5/3} \frac{\Delta\lambda_i}{N_i{}^2} | \tag{17}$$

$$i = 2, \ldots, n$$

A Closer Look at Eq. (17)

It is now desirable to examine the A and B terms of Eq. (17). In doing this, use will be made of the approximation

$$(1 \pm b)^m \cong 1 \pm mb \tag{18}$$

which holds when b is small compared to one. It can be shown that using Eq. (18) in Eq. (17) results in the following coefficient of the $\Delta\lambda_i / N_i{}^2$ term:

$$(1 - \frac{\Delta\lambda_i}{1080\, N_i})(1 + \frac{\Delta\lambda_i}{216\, N_i})$$
$$= 1 + \frac{\Delta\lambda_i}{180\, N_i} - \frac{\Delta\lambda_i{}^2}{233{,}280\, N_i{}^2} \tag{19}$$

For $\Delta\lambda_i / 180\, N_i$ sufficiently small, which is usually the case, the coefficient of the $\Delta\lambda_i / N_i{}^2$ term is close to one. Returning now to Eq. (17), it can be seen that the equation can be approximated by

$$N_1 / N_i = | \Delta\lambda_1 / \Delta\lambda_i |^{1/2} \qquad i = 2, \ldots, n \tag{20}$$

Use of Eq.(20)

Equation (20) is used with the requirement that the sum of the N_i must be a preselected value to provide optimal values of the N_i. These will almost always be nonintegral values. Since it is necessary that the N_i be integers, it is appropriate to round the results of Eq. (20) to the nearest integers. The results of the rounding process are almost always the optimal practicable results. It is explained later that, in some situations, the rule of rounding to the nearest integers must be modified, and simple rules are given for these situations.

Sample Application

Reference 2 contains a description of a particular case of on-orbit servicing and a trial-and-success method for obtaining the optimal distribution of transfer ellipses. The case is described as follows: a) four satellites in synchronous equatorial orbit; b) intersatellite separations of 35, 150, and 40 deg; and c) total transfer time allowable = 11.375 days (total number of revolutions = 12). Using Eq. (20) leads to the following results:

$$\frac{N_1}{N_2} = | \frac{35}{150} |^{1/2} = 0.4830; \quad \frac{N_1{}^{\dagger}}{N_3} = | \frac{35}{40} |^{1/2} = 0.9354 \tag{21}$$

Since

$$N_1 + N_2 + N_3 = 12 \tag{22}$$

it follows that

$$N_1 + \frac{N_1}{0.4830} + \frac{N_1}{0.9354} = 12 \tag{23a}$$

† $N_2 / N_3 = | 150/40 |^{1/2}$ could have been used here as well. The choice is arbitrary.

$$N_1 = \frac{12}{4.139} = 2.90 \cong 3 \tag{23b}$$

$$N_2 = \frac{N_1}{0.4830} = 6.00 \tag{23c}$$

$$N_3 = 12 - 3 - 6 = 3 \tag{23d}$$

The optimal distribution of the total of 12 transfer ellipse revolutions has thus been found to be 3,6,3. This is the same answer as found by the trial-and-success technique in Ref. 2.

What to Do When Ambiguous Results Occur

Example

Occasionally, use of Eq. (20) will yield results such as in the following example:

The satellite central angle separations are:

$$\Delta\lambda_1 = 85°, \quad \Delta\lambda_2 = 120°, \quad \Delta\lambda_3 = 55° \tag{24}$$

The required total number of intersatellite transfer ellipse revolutions is

$$N_T = 16 \tag{25}$$

Use of Eq. (20) gives the following:

$$\begin{aligned} N_1 &= 5.346 \cong 5 \\ N_2 &= 6.353 \cong 6 \\ N_3 &= 4.301 \cong \underline{4} \\ & \qquad\quad 15 \end{aligned} \tag{26}$$

Rounding to the nearest integers results in an N_i total of 15 instead of the required 16. The question is: What should be done?

It has been found in this and similar cases that rounding up the particular N_j that is closest to the next higher digit gives best results. That is, in the case being considered, N_2 should be made equal to 7 because it is closer to 7 than N_1 is to 6 and N_3 is to 5. It is shown below that the rounding up of N_2 in this example yields optimal results. Using the $\Delta\lambda$'s of Eqs. (24) in Eqs. (9) and (10) gives the results shown in Table 1.

Table 1 Effect of ellipse distribution on total ΔV

Distribution of transfer ellipses	Total ΔV, fps
5, 7, 4	937
6, 6, 4	938
5, 6, 5	941

As predicted, the (5, 7, 4) combination is the optimum. (It is also clear that rounding up any one of the three N_i is satisfactory in this case because the ΔV spread is very small. It is often found, as it has been here, that off-optimal selections of the N_i are almost as good as the optimal. This occurs when the N_i are large, as in this example.)

A Particular Type of Case

Consider a case in which $(n+1)$ satellites are uniformly distributed over $\Delta\lambda$ degrees of central angle. Each intersatellite central angle in such a case is equal to each of the others. In attempting to determine an optimal distribution of N_T transfer ellipse revolutions among the n intersatellite transfers, it seems reasonable that each transfer should require the same number of ellipses as each of the others. A problem arises, however, whenever $N_T \neq cn$, $c = 1,2,3...$ That is, in a case in which the required total number of transfer ellipses is not equal to an integral multiple of the number of transfers, the intersatellite transfers cannot be equal even though, ideally, they should be. An example will illustrate this situation.

Let four satellites be separated from one another by 60° (each pair), and let the required total number of transfer ellipses be 16. Using Eq. (20),

$$\begin{aligned} N_1 + N_2 + N_3 &= 16 \\ N_1 = N_2 &= N_3 \end{aligned} \tag{27}$$

Then,

$$N_1 + N_1 + N_1 = 16 \tag{28}$$

$$N_1 = 5.33 \quad \cong 5 \tag{29a}$$

$$N_2 = N_1 = 5.33 \cong 5 \tag{29b}$$

$$N_3 = N_1 = 5.33 \cong 5 \tag{29c}$$

Obviously, this solution is unsatisfactory because the sum of the N_i is not equal to the required total. One of the transfers must be given one more ellipse. Because each of the intersatellite separations is the same as each of the others, the extra ellipse must be added arbitrarily. The total required ΔV for all of the transfers will be the same whether the extra ellipse is added to the first, second, or third transfer.

This particular type of case is discussed here to illustrate a point. Users of the method discussed in this paper may be tempted to improve the method by redoing the derivation with fewer approximations. The objective in redoing the derivation would be to discover a new set of equations that avoids ambiguities of the kind discussed at the beginning of this section. However, the example just given shows that certain cases will always be ambiguous. No equations, elaborate or simple, will circumvent the situation. Therefore, the equations (and simple rules) given in this paper are sufficient for all cases.

The Simple Rules

Below are the rules to follow when the optimization technique described in this paper results in an N_i total different from that required. The rules are not derived or proved in this paper.

When ΣN_i is too small, round up the N_j that is closest to the higher integer. If ΣN_i is still too small, round up the N_j that is second-closest to the higher integer. Repeat, if necessary, until ΣN_i has the required value.

When ΣN_i is too large, round down the N_j that is closest to the lower integer. If ΣN_i is still too large, round down the N_j that is second-closest to the lower integer. Repeat, if necessary, until ΣN_i has the required value.

When pairs of the N_i have values exactly midway between two integers (such as $N_j = X.5$, $N_k = Y.5$), round either of the two up and the other down. The same total ΔV requirement is produced in either case.

When all of the N_i are equal, round any of them up or down as required until ΣN_i has the required value.

Conclusions

The distribution of intersatellite transfer ellipses given by Eq. (20), as augmented by the simple rules given in the preceding section, is optimal when minimization of total ΔV is the criterion for optimality. No other distribution is superior.

Appendix

Alternative Expressions for ΔV

$$\Delta V = 2V \left| \left[2 - \left(1 - \frac{\Delta\lambda}{360\,N}\right)^{-2/3} \right]^{1/2} - 1 \right| \tag{A1}$$

$$\Delta V = 2V \left| \left[2 - \left(1 - \frac{\tau \Delta\lambda}{360\, \Delta T} \right)^{2/3} \right]^{1/2} - 1 \right| \tag{A2}$$

$$\Delta V = 2V \left| \left[2 - \left(\frac{\tau_i}{\tau} \right)^{-2/3} \right]^{1/2} - 1 \right| \tag{A3}$$

References

[1] Jensen, J., Townsend, G., Kork, J., and Kraft, D., *Design Guide to Orbital Flight,* McGraw-Hill, New York, 1962.

[2] Fallin, E. H., III, "Time Requirements for Multiple Intersatellite Transfers," *Journal of Spacecraft and Rockets,* Vol. 11, June 1974, pp. 445-446.

Analysis of Thick Cylinders with Internal Semicircular Grooves Subjected to Internal Pressure

Mihir Kumar Jana* and G. Venkateswara Rao†
Vikram Sarabhai Space Centre, Trivandrum, India

Introduction

SOLID propellant rocket motor grains are generally thick cylinders with internal axisymmetric grooves. Certain analysis and design modifications to reduce the effects of such grooves were investigated in Refs. 1 and 2. Essentially, the propellant grain is a viscoelastic material and a rigorous viscoelastic analysis is to be performed to assess the performance of the grain. Since the numerical procedure involved in the viscoelastic analysis is a series of elastic analyses in the time domain, a very accurate and reliable elastic analysis tool is necessary to get a satisfactory viscoelastic solution. And for configuration involving internal grooves, the continuum analysis becomes tedious and finite element methods can be advantageously used to get accurate stress pictures around the grooves. For this purpose, a six-degree-of-freedom triangular solid ring element is available in the literature,[3] but it is the authors' experience that the use of this element does not give satisfactory stress picture around the grooves even with a very fine mesh division. Then the alternative is to use the complex isoparametric elements[3] or high-precision triangular solid axisymmetric ring element,[4] to get a realistic solution around the grooves.

The main aim of the present Note is to get the elastic stress distribution using the high-precision triangular ring element[4] along the semi-circular internal grooves of a thick cylinder subjected to internal pressure, which is a typical sample of a rocket grain, for various groove geometries. As the emphasis given in the Note is the study of the variation of stress concentration along the groove, a typical configuration of a thick cylinder with internal grooves is considered and groove geometries alone are varied.

High Precision Finite Element

A typical ring element with triangular cross section is considered for the analysis of axisymmetric solids. The element has three nodes and six degrees of freedom per node, namely, u, u_x, u_y, v, v_x, v_y, where subscript denotes partial differentiation. The integration involved in the derivation of element matrices are evaluated numerically using the algorithm given in Ref. 5. The detailed derivation of element stiffness matrix and the consistent load vectors of the high-precision element are presented in Ref. 4 and hence these details are not repeated here. A significant advantage of the present element over the element given in Ref. 3 is that all the strain components are included in the nodal degrees-of freedom and hence the evaluation of nodal strains and stresses is extremely simple.

To demonstrate the extreme accuracy and reliability of the present high-precision element, it is first applied to the standard Lame's thick cylinder problem and the results are presented in Table 1. It is evident from this table that the results obtained by the present element are extremely accurate even with a very coarse four element idealization.

Secondly, the problem of a shaft in tension with a deep hyperbolic notch (see Fig. 1) is considered to establish the efficiency of the present problem in predicting the stress concentration. The tangential stress at the base of the notch is calculated for three different sets of idealizations: a) 31 elements and 25 nodes; b) 31 elements and 25 nodes taking finer mesh around the notch; and c) 74 elements and 49 nodes. The stress concentration factor at the base of the notch for these three cases are found to be 2.771, 2.5887, and 2.3446,

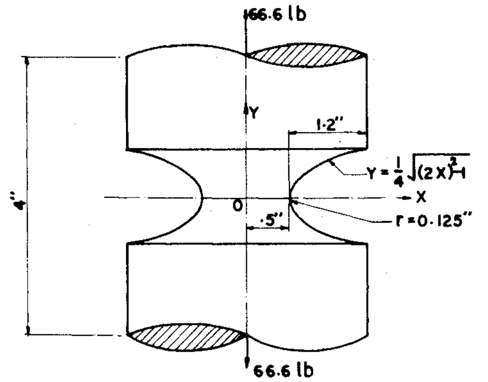

Fig. 1 Circular shaft under tension with deep hyperbolic notch.

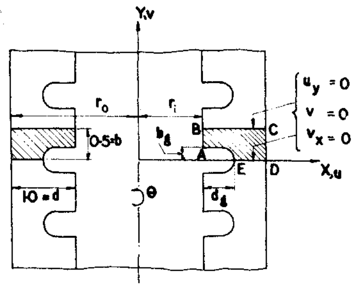

Fig. 2 Long, thick cylinder with equally spaced semi-circular grooves.

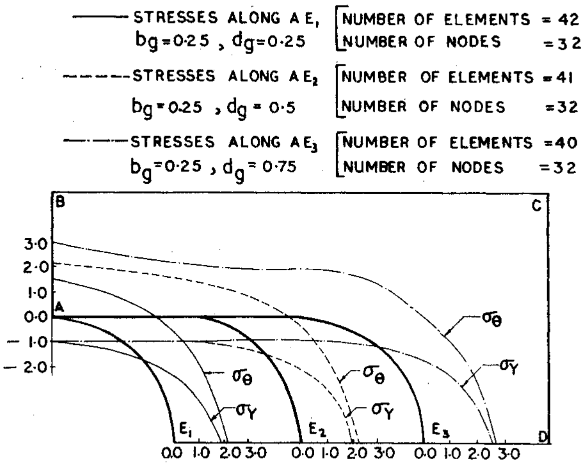

Fig. 3 Stress distribution along AE_1, AE_2 and AE_3.

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*Structural Engineer. Structural Engineering Division.